Runge-Kutta Methods

EE 451-Introduction to Robot Control

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ODE's

$$\frac{dy}{dx} = f(x, y)$$

Solution to the ODE's in general:

$$y_{i+1} = y_i + \phi(x_i, y_i, h)h$$

- h: step-size
- $\phi(x_i, y_i, h)$: increment function (slope over the interval). In general

$$\phi = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n$$

ODE's

$$\phi = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n$$

- a's are constant
- k's are recurrence relationships

$$\begin{aligned}
k_1 &= f(x_i, y_i) \\
k_2 &= f(x_i + p_1 h, y_i + q_{11} k_1 h) \\
k_3 &= f(x_i + p_2 h, y_i + q_{21} k_1 h + q_{11} k_2 h) \\
&\vdots \\
k_n &= f(x_i + p_{n-1} h, y_i + q_{n-1,1} k_1 h + \cdots + q_{n-1,n-1}, k_{n-1} h)
\end{aligned}$$

where p's and q's are constant.

Second-Order Runge Kutta Methods

- $y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$
- $$\begin{aligned} k_1 &= f(x_i, y_i) \\ k_2 &= f(x_i + p_1 h, y_i + q_{11} k_1 h) \end{aligned}$$

Second-Order Runge Kutta Methods

To find a_1, a_2, k_1, and k_2, Taylor series expansion for y_{i+1}

$$\begin{aligned} y_{i+1} &= y_i + f(x_i, y_i)h + \frac{f'(x_i, y_i)h^2}{2!} \\ &= y_i + f(x_i, y_i)h + \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right)\frac{h^2}{2!} \end{aligned}$$

Remember, Taylor series for 2-variable function

$$f(x_i + p_1 h, y_i + q_{11} k_1 h) = f(x_i, y_i) + p_1 h \frac{\partial f}{\partial x} + q_{11} k_1 h \frac{\partial f}{\partial y} + O(h^2)$$

Rewrite y_{i+1} as

$$\begin{aligned} y_{i+1} &= y_i + \left[a_1 f(x_i, y_i) + a_2 f(x_i, y_i)\right] h \\ &+ \left[a_2 p_1 \frac{\partial f}{\partial x} + a_2 q_{11} f(x_i, y_i) \frac{\partial f}{\partial y}\right] h^2 + O(h^3) \end{aligned}$$

Fourth-Order Runge-Kutta Methods

- Most popular - classical fourth-order RK method
- $y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$ where

$$\begin{aligned}
k_1 &= f(x_i, y_i) \\
k_2 &= f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h) \\
k_3 &= f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h) \\
k_4 &= f(x_i + h, y_i + k_3 h)
\end{aligned}$$

Fourth-Order Runge-Kutta Methods

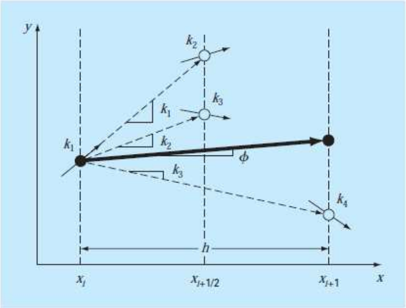

Figure : Graphical representation of the slope estimates of fourth-order RK method

Systems of Equations

$$
\begin{aligned}
\frac{dy_1}{dx} &= f_1(x, y_1, \ldots, y_n) \\
&\vdots \\
\frac{dy_n}{dx} &= f_n(x, y_1 \ldots, y_n)
\end{aligned}
$$

Modify the previous algorithms such that your code includes

- n equations
- n initial consitions
- a loop for computing a new value for each dependent variable

and computes slopes for each independent variable.

Pseudo Code for Runge-Kutta Methods

```
sub RK4(x, y, h, ynew)
    CALL derivs(x, y, k1)
    ym = y + k1*h/2
    CALL derivs(x+h/2, ym, k2)
    ym = y + k2*h/2
    CALL derivs(x+h/2, ym, k3)
    ye = y + k3*h
    CALL derivs(x+h, ye, k4)
    slope = (k1 + 2*(k2 + k3) + k4)/6
    ynew = y + slope*h
    x = x + h
END sub
```