EE 653 - Fundamental Concepts

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Problem Formulation
- Optimal Control Problem

Mathematical Model

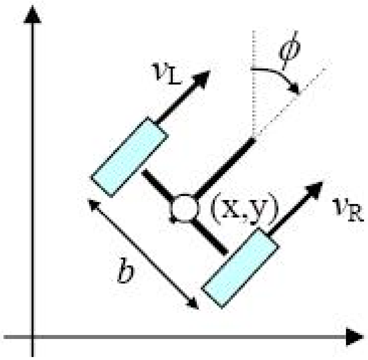
Process to be controlled

- State variables or states: $x_i(t) \in R$, $i = 1, \ldots, n$ or $x(t) \in R^n$, $t \geq 0$ 0
- Control inputs or control: $u_i(t) \in R$, $i = 1, \ldots, m$ or $u(t) \in R^m$, $t \geq 0$ 0
- **System dynamics (General form: First-order)**

$$\begin{aligned} \dot{x}_1(t) &= a_1(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), t) \\ \vdots &= \vdots \\ \dot{x}_n(t) &= a_n(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), t) \end{aligned}$$

or simply $\dot{x}(t) = a(x(t), u(t), t)$

Modeling Example - Robot with Differential Movement



Assumptions

- Movement on planar surface with z-axis perpendicular to floor
- No slippage
- Rigid robot links
- For small Δt, change in orientation $\Delta\phi$ is very small.

Model Formulation

- Seperation of two wheels b
- Wheels with radius r
- Angular velocities of right and left wheels – ω_r and ω_l -
- Linear velocities of right and left wheels $e_1 v_r$ and $e_1 v_l$
- Note $v_r = r \times \omega_r$ and $v_l = r \times \omega_r$
- Linear velocity of mid point $u = \frac{1}{2}(v_r + v_l)$
- Angular velocity of midpoint $\omega = \frac{1}{b}(v_r - v_l) = \frac{r}{b}(\omega_r - \omega_l)$
- Path curvature $\kappa = \frac{2}{b}\frac{v_r - v_l}{v_r + v_l}$
- If $v_l = -v_r$, $\kappa \to \infty$ - Turn on the spot
- In order to minimize stress on mechanical structures, avoid commanding wheel speeds of different sign
- Note that $v_l, v_r \in [0, V_m]$ $\kappa \in \left[-\frac{1}{b}, \frac{1}{b}\right]$ and $u \leq \frac{V_m}{2}$

Robot Model

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} -\frac{r}{2}\sin(\phi) & -\frac{r}{2}\sin(\phi) \\ \frac{r}{2}\cos(\phi) & \frac{r}{2}\cos(\phi) \\ -\frac{r}{b} & -\frac{r}{b} \end{bmatrix} \begin{bmatrix} \omega_r \\ \omega_l \end{bmatrix}$$

- Identify: State related n and x, Control input related m and u
- Simulate - initial at origin with heading upwards.

$$\omega_1(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 1 & 1 \leq 4 \\ -t+5 & 4 \leq t \leq 5 \end{cases}, \omega_2(t) = \omega_1(t)$$

$$\omega_1(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 1 & 1 \leq 4 \\ -t+5 & 4 \leq t \leq 5 \end{cases}, \omega_2(t) = -\omega_1(t)$$

$$\omega_1(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 1 & 1 \leq 4 \\ -t+5 & 4 \leq t \leq 5 \end{cases}, \omega_2(t) = \begin{cases} 2t & 0 \leq t \leq 1 \\ 2 & 1 \leq 4 \\ -2t+5 & 4 \leq t \leq 5 \end{cases},$$

State and Control Functions

- State $x : R \to R^n$
- Consider $\mathcal{X}$ - The space of states
- Control input $u : R \to R^m$
- Consider $\mathcal{U}$ - The space of control inputs
- Generalization to higher dimensions on the domain

Related Issues

- Controllability
- Observability
- Uniqueness
- Global optimality
- Consistency of constraints

Performance

- Objective $J(u) = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), u(t), t)dt$
- $h = 9,\ g = 1$
- $h = 0,\ g = u(t)^T u(t) = u^T u.$
- $g = 0,\ h = (x(t_f) - x_g)^T (x(t_f) - x_g)$

Problem Statement

Given a process model $\dot{x} = a(x(t), u(t), t)$

Find $u^* \in \mathcal{U}$ and $x^* \in \mathcal{X}$ s.t

$u^* \in \arg\min_{u \in \mathcal{U}} \; J(u)$