EE 653 - Extrema and Calculus of Variations Concepts

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Optimization - Extrema Concepts
- Preliminaries
- Optimality

Function vs Functional

• Function: A map that maps each element in its domain D to a value in its range $\mathcal{R}$

- $D = R$, $\mathcal{R} = R^n$
- $D = R \times R$, $\mathcal{R} = R^n$

• Function Spaces: $\mathcal{X}$ - Space of functions

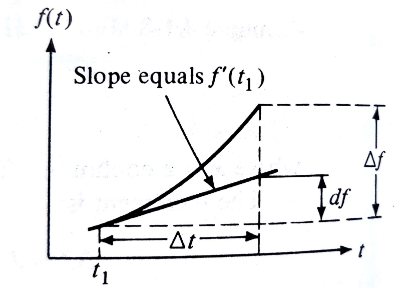
• Functional: A map from a function space $\mathcal{X}$ to a real number in R.

- $J(x) = \int_{t_0}^{t_f} x^2(t)dt$

Increment of Functions

Differential of a function

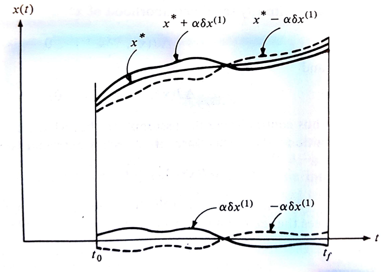
$$\underbrace{f(q+\delta q) - f(q)}_{\text{Increment}} = \underbrace{D_q f(q)^T \delta q}_{df(q,\delta q)} + O(\|\delta q\|^2)$$



$$\Delta f(q, \delta q) = f(q+\delta q) - f(q)$$

Increment of Functionals

$$\Delta J(x, \delta x) = J(x + \delta x) - J(x)$$



Variation of a Functional:

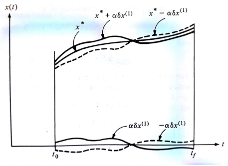
$$\underbrace{J(x + \delta x) - J(x)}_{\text{Increment}} = \underbrace{\delta J(x, \delta x)}_{\text{Variation}} + O(\|\delta x\|^2)$$

Optimality of Functions

- A function $f : \mathcal{D} \rightarrow R$ has a relative optimum of $\exists q \in \mathcal{D}$ such that $\forall q' \in N_\epsilon(q)$ where $q' = q + \delta q$
  - $\Delta f(q, \delta q) \geq 0 \rightarrow$ Local minimum
  - $\Delta f(q, \delta q) \leq 0 \rightarrow$ Local maximum
- Necessary condition for optimality $D_q f(q) = 0$

Calculus of Variations - Optimality of Functionals

- A functional $J : \mathcal{X} \rightarrow R$ has a relative optimum at $x \in \mathcal{X}$ if $\exists \epsilon > 0$ such that $\forall x' \in N_\epsilon(x)$ where $x' = x + \delta x$
  - $\Delta J(x, \delta x) \geq 0 \rightarrow$ Local minimum
  - $\Delta J(x, \delta x) \leq 0 \rightarrow$ Local maximum



- **Fundamental Theorem of Calculus of Variations:**

▷ If x is an extremal function, then $\forall \delta x$, $\delta J(x, \delta x) = 0$