EE 653 - Introduction & Review

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Scalar Differential Systems

$x(t) \in R, \; t \geq 0$

$$\begin{aligned} \dot{x} &= f(x,t) \\ \dot{x} &= f(x) \end{aligned}$$

- Closed-form solutions
  - $\dot{x} = ax \qquad a \in R$
  - $\dot{x} = ax + bu \qquad a, b \in R, \; u(t) \in R$ given
- Numerical solutions

Taylor Expansion

Let $f : R \to R$.

- Taylor expansion around $x_0 \in R$.

1. Taylor expand $cos(x)$ around $x_0 = 0$.
2. Taylor expand e^x around $x = 0$.

$\to$ Analytic function

Matrix Properties

A, D - $n \times n$ matrix, $x \in R^n$

Transpose

- $(AD)^T$
- $(x^T A x)^T$

Positive definite, Negative definite

Positive semidefinite, Negative semidefinite

Eigenvalues, eigenvectors

- Consider $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Compute eigenvalues and eigenvectors.

General Differentiation

$x(t) \in R^n$, A - $n \times n$ matrix

Consider $f : R^n \to R$. The gradient is $D_x f(x)$ Let

- $f(x) = x^T A x$, compute $D_x f(x)$

Hessian

Consider $f : R^n \to R^m$. The Jacobian is $D_x f(x)$

- If $f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}$, compute $D_x f(x)$
- $f(x) = Ax$, compute $D_x f(x)$

Chain Rule

Let $g : R^n \to R^m$, $f : R^m \to R$

Consider $f \circ g$ and compute $D_x f \circ g$ using chain rule

Taylor Expansion of Scalar Functions with Vector Variables

Let $f : R^n \rightarrow R$.

- Taylor expansion around $x_0 \in R^n$.

Matrix Differential Systems

$x(t) \in R^n,\ u(t) \in R^m,\ t \geq 0$

$$\begin{aligned} \dot{x} &= f(x,t) \\ \dot{x} &= f(x) \end{aligned}$$

- Closed-form solutions. Let A - $n \times n$ matrix
  - $\dot{x} = Ax$,
  - $\dot{x} = Ax + Bu$,
- Numerical solutions

Cayley-Hamilton Theorem

A - $n \times n$ matrix

Characteristic polynomial

Cayley-Hamilton Theorem

Using Cayley-Hamilton Theorem to Compute Analytic Matrix Functions

- $f(s) = \sum_{k=1}^{\infty} \alpha_k s^k$
- Consider A - $n \times n$ matrix with $\Delta(s)$ and eigenvalues λ_i.
- $f(s) = \Delta(s)Q(s) + R(s)$ where $R(s)$ is of degree $n-1$ or less
- $R(\lambda_i) = \sum_{k=0}^{n-1} \gamma_k \lambda_i^k = f(\lambda_i)$ (Defines a set of equations for solving γ_k)
- $f(A) = R(A)$
  - Consider $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, compute e^A, e^{At}, $sin(A)$, $sin(At)$